Pergamon

Int. J. Heat Mass Transfer. Vol. 41, No. 11, pp. 1531–1541, 1998

Printed in Great Britain
0017–9310/98 $19.00+0.00

PII: S0017–9310(97)00214–7

Influence of radial seepage on temperature distribution around a cylindrical cavity in a porous medium

JIM BROWN,† ALAN VARDY and ZHAOYANG ZENG
Civil Engineering Department, University of Dundee, Dundee DD1 4HN, U.K.

(*Received* 14 *October* 1996 *and in final form* 14 *July* 1997)

Abstract—Temperature distributions are obtained around a long cylindrical cavity in a permeable medium in the presence of steady radial seepage. Constant temperature and constant heat flux conditions are considered at the cavity surface. Transient solutions are derived as quadratures, yielding closed form solutions for particular seepage rates. Steady states are derived as limiting cases. Small and large time approximations are derived for the cavity temperature and flux. With inward seepage, the thermal radius of influence is restricted and steady state attained more rapidly. For outward seepage, the radius of influence becomes infinite and the steady temperatures uniform.

1. INTRODUCTION

Heat conduction in a porous medium through which fluid is flowing is of relevance to thermal storage systems, geothermal energy production systems, reaction chambers, unlined tunnels, mining, etc.—see, for example, Gringarten *et al.* [1], Claesson and Dunand [2], Iguchi [3], McPherson [4], Kimura *et al.* [5]. In all cases, convection of thermal energy by the fluid flow can have a considerable influence on the overall behaviour of the heated zone.

In general, the fluid flow might be caused by either natural or impressed pressure distributions. In low-permeability media experiencing small temperature differences, however, buoyancy-induced flows (see for example, Bau [6]) are usually small in comparison with pressure-induced flows. These are the conditions assumed herein. They often exist in the ground surrounding mines and tunnels, for instance, because the opening creates a local variation in piezometric pressure (in comparison with the surrounding medium).

The conduction of heat in an infinite medium bounded internally by a cylindrical cavity was first considered by Nicolson [7] using a Green's function approach to solve the case of constant cavity surface temperature. Goldstein [8] applied the Heaviside operator method to this and other cases. Carslaw and Jaeger [9, 10] developed solutions using the Laplace transform technique, summarized the solutions for the standard steady and periodic boundary conditions and presented a solution for the temperature distribution due to constant cavity surface temperature in the presence of axisymmetric, radially outwards, fluid flow.

The following theoretical development concerns the particular case of a long cylindrical cavity in an infinite homogeneous porous medium. Transient heat transfer is considered, but the fluid flow (seepage) is assumed constant because the impressed piezometric head difference is constant. The solution in ref. [10] is extended to the case of radially inwards flow, and the conduction heat flux at the cavity is found. Solutions are also obtained for the temperature distribution induced by a constant prescribed cavity surface heat flux due to conduction.

For transient conditions, solutions are obtained using Laplace transforms and are expressed in the form of integrals of Bessel functions in which the order of the Bessel functions depends on the rate of seepage. In some particular cases, closed-form solutions can be obtained. In most cases, however, the integrals must be evaluated numerically. Exact steady-state solutions are obtained as limiting cases. In addition, some approximate closed-form solutions for small and large times are obtained by approximating the Bessel functions before carrying out the inverse Laplace transform.

2. MATHEMATICAL FORMULATION

2.1. *Introduction*

Consider a long horizontal cylindrical cavity (Fig. 1) of radius a in an infinite region filled with a saturated porous medium of uniform permeability K. The permeability and seepage velocity are stipulated to be sufficiently small for the temperature T of the fluid and the matrix at any point to be equal at all times. The fluid and the porous matrix are assumed incompressible and seepage velocities due to natural convection are assumed to be negligible.

† Author to whom correspondence should be addressed.

NOMENCLATURE

a	radius of cavity [m]
A, A_1	constants of integration
b	order of Bessel function in equation (17)
B, B_1	constants of integration
C_P	heat capacity at constant pressure [J kg^{-1} K^{-1}]
erfc(x)	complementary error function
$\bar{F}(s)$	Laplace transform of $F(t)$; see equation (10)
g	gravitational acceleration [m s^{-2}]
G	non-dimensional steady state cavity heat flux (equation 30)
H, H_0	piezometric head, piezometric head at radius r_0 [m]
$I_\lambda(x)$	modified Bessel function of first kind, order λ
$J_\lambda(x)$	Bessel function of first kind, order λ
k	thermal conductivity [W m^{-1} K^{-1}]
K	permeability [m^2]
$K_\lambda(x)$	modified Bessel function of second kind, order λ
p	pressure [Pa]
q	radial heat flux (positive = outwards) [W m^{-2}]
Q_F	radial fluid flow rate per unit length of cavity (positive = outwards) [m^2 s^{-1}]
Q_{FN}	radial inward fluid flow rate per unit length of cavity [m^2 s^{-1}]
Q_h	heat flow rate per unit length of cavity [W m^{-1}]
r, r_0	radial distance, radius of influence [m]
R	non-dimensional radial distance $\equiv r/a$
R^*	non-dimensional quantity $\equiv (R-1)/2\sqrt{\tau}$
s	Laplace transform variable; see equation (10)
t	time [s]
T	(absolute) temperature [K]
T_0	initial temperature [K]
v_r	radial seepage velocity [m s^{-1}]
V	expression defined in equation (38)
w	integration variable; e.g. equation (22)
x	arbitrary variable
$Y_\lambda(x)$	Bessel function of second kind, order λ
z	axial coordinate [m]
Z	height above a datum.

Greek symbols

α	effective thermal diffusivity of saturated medium [m^2 s^{-1}]
β	$\sqrt{(\sigma s/\alpha)}$; see equation (14)
$\Gamma(x)$	gamma function
ε	parameter in the Bromwich integral equation (16)
ζ	grouping in equation (50)
η	grouping in equation (56)
κ	forced convection parameter $Q_F/4\pi\alpha$
θ	azimuthal coordinate
λ	inward flow parameter $Q_{FN}/4\pi\alpha$
μ	dynamic viscosity [Pa s]
ρ	fluid density [kg m^{-3}]
σ	ratio of heat capacities of saturated medium and fluid
τ	non-dimensional time $\equiv \alpha t/a^2$.

Subscripts

a, ∞	quantity at radius a, at infinity
H	prescribed heat flux at the cavity surface
T	prescribed temperature at the cavity surface.

Superscripts

*	non-dimensional variable
—	an 'over bar' denotes a Laplace transformed variable.

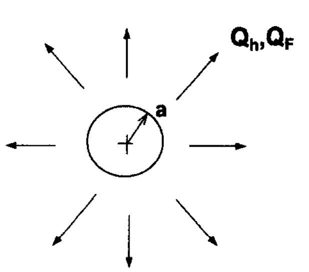


Fig. 1. Cross-section of cavity showing positive directions of heat and fluid flows.

The piezometric head H in the fluid is:

$$H = \frac{p}{\rho g} + Z \tag{1}$$

where ρ is the fluid density, g is the gravitational constant and Z is height above a datum. The piezometric head is assumed to have the uniform value H_∞ remote from the cavity and the uniform value H_a around the cavity surface. Since the seepage rates are assumed to be small, the consequent axial rates of flow and the associated axial piezometric head gradient in the cylindrical cavity will also be small. In these circumstances, it is reasonable to assume that the variation of seepage velocity and temperature with respect to the axial direction will be small enough to be neglected.

The temperature is assumed to be initially uniform throughout the medium with the value T_0. Conditions of either prescribed temperature T_a, or of prescribed surface heat flux q_a due to conduction, are imposed uniformly over the cavity surface at time $t = 0$ and are maintained constant thereafter. With these conditions, it is reasonable to expect that the fluid and heat flows will be axisymmetric as well as independent of distance along the cavity axis. Accordingly, a cylindrical polar coordinate system r, θ, z is chosen, with $r = a$ defining the position of the cavity surface.

The governing equations for convection in a porous medium may be found, for example, in Cheng [11], Poulikakos and Bejan [12], Nield and Bejan [13]. In general, these comprise the equations of conservation of mass, momentum (Darcy equation) and energy, and an equation of state. However, since both buoyancy and fluid and matrix compressibility effects are to be neglected, the last of these equations is not required. Furthermore, the equations of conservation of mass and momentum are uncoupled from the energy equation and can thus be integrated separately from it.

2.2. *Fluid flow*

In cylindrical coordinates, the equations of conservation of mass and momentum for steady fluid flow take the form:

$$\frac{1}{r}\frac{\mathrm{d}(rv_r)}{\mathrm{d}r} = 0 \tag{2}$$

$$v_r = -\frac{K\rho g}{\mu}\frac{\mathrm{d}H}{\mathrm{d}r} \tag{3}$$

where v_r and μ are the radial seepage velocity and dynamic viscosity of the fluid, respectively. Only steady seepage is considered because the boundary conditions are assumed steady and the matrix and fluid are assumed incompressible. Any transient state would therefore be very short in comparison with heat conduction time scales.

Equation (2) may be integrated directly to give the radial seepage velocity as:

$$v_r = \frac{Q_F}{2\pi r} \tag{4}$$

where Q_F is the volumetric seepage rate per unit length out of the cavity.

By eliminating v_r from equations (3) and (4), and integrating from $r = a$ (the cavity surface) to radius $r = r_0$ (the effective radius of influence of the cavity) where the piezometric head is H_0, the seepage rate may be shown to be:

$$Q_F = \frac{2\pi K\rho g(H_a - H_0)}{\mu \ln(r_0/a)}. \tag{5}$$

When $H_a > H_0$, the fluid flows radially outwards; when $H_a < H_0$, it flows radially inwards. Thus, with the assumptions of steady uniform piezometric head at $r = a$ and $r = r_0$, the radial seepage essentially becomes a parameter in the heat flow problem.

2.3. *Temperatures and heat flux*

The energy equation may be expressed as:

$$\sigma\frac{\partial T}{\partial t} + v_r\frac{\partial T}{\partial r} = \alpha\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) \tag{6}$$

in which σ is the ratio of the heat capacities of the saturated medium and the fluid and α is the effective thermal diffusivity of the saturated porous medium. This is defined (e.g. ref. [11]) as:

$$\alpha \equiv \frac{k}{\rho C_P} \tag{7}$$

where k and C_P are the thermal conductivity of the saturated medium and the specific heat capacity of the fluid, respectively.

Since the temperature T appears only in derivatives in equation (6), it is determined only to an arbitrary constant, which is taken herein to be an initial, uniform temperature T_0 existing before any thermal disturbance of the system. Then, eliminating v_r between equations (4) and (6), the governing differential equaton for the variation of temperature is:

$$\sigma\frac{\partial T}{\partial t} + \frac{Q_F}{2\pi r}\frac{\partial T}{\partial r} = \alpha\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) \tag{8}$$

which may be integrated for appropriate initial and boundary conditions.

The heat flux q due to conduction at any radius in the saturated medium is:

$$q = -k\frac{\partial T}{\partial r}. \tag{9}$$

2.4. *Initial conditions and boundary conditions*

Herein, the phenomena under study are the changing thermal conditions during the transition from one steady state condition to another. The rate of fluid flow remains steady, but the temperature field evolves in response to a step change in the prescribed conditions at the cavity surface.

When the temperature is stipulated at the cavity surface, the prescribed conditions may be stated as:

Initial conditions ($t < 0$): $T(r) = T_0$

Boundary conditions ($t \geqslant 0$):

$$T_{r=a} \equiv T_a; \quad T(r_0) = T_0.$$

When the heat flux in the solid matrix is stipulated at the cavity surface, the prescribed conditions may be stated as:

Initial conditions ($t < 0$): $T(r) = T_0$

Boundary conditions ($t \geqslant 0$):

$$q_{r=a} \equiv q_a; \quad T(r_0) = T_0.$$

3. GENERAL SOLUTION IN THE LAPLACE TRANSFORM DOMAIN

The Laplace transform $\bar{F}(s)$ of any time-dependent function $F(t)$ is defined by:

$$\bar{F}(s) = \int_0^\infty e^{-st} F(t) \, dt \tag{10}$$

in which the Laplace parameter s may be chosen arbitrarily. By transforming the temperature T in this manner and using $T = T_0$ at $t = 0$, equation (8) may be expressed as:

$$\frac{d^2 \bar{T}}{dr^2} + \left(1 - \frac{Q_F}{2\pi\alpha}\right) \frac{1}{r} \frac{d\bar{T}}{dr} - \frac{\sigma s}{\alpha} \bar{T} = -\frac{\sigma}{\alpha} T_0 \tag{11}$$

which is an ordinary differential equation in the independent variable r. The homogeneous form of equation (11) has the form of the modified Bessel equation. The general solution of this together with the particular solution T_0/s is:

$$\bar{T} = [A_1 I_\kappa(r\beta) + B_1 K_\kappa(r\beta)] r^\kappa + T_0/s \tag{12}$$

where $I_\kappa(x)$, $K_\kappa(x)$ are modified Bessel functions of the first and second kind and of order equal to the forced convection parameter κ, defined as

$$\kappa \equiv \frac{Q_F}{4\pi\alpha} \tag{13}$$

in which $Q_F > 0$ corresponds to outward fluid flow. The parameter β satisfies:

$$\beta^2 = \sigma s/\alpha. \tag{14}$$

The constants of integration A_1 and B_1 are determined from conditions at infinity and at the cavity surface. The first of these yields $A_1 = 0$ because $I_\kappa(r\beta) \to \infty$ as $r\beta \to \infty$. Thus, equation (12) reduces to:

$$\bar{T} = B_1 K_\kappa(r\beta) r^\kappa + T_0/s. \tag{15}$$

The constant B_1 is to be found from the cavity surface boundary condition. The temperature distribution is then obtained by performing the inverse Laplace transform using the Bromwich integral:

$$F(t) = \frac{1}{2\pi i} \int_{\varepsilon - i\infty}^{\varepsilon + i\infty} e^{st} \bar{F}(s) \, ds \tag{16}$$

in which ε is chosen such that all singularities of the integrand lie to the left of the line $s = \varepsilon$.

In considering the properties of equation (15) it should be noted that:

$$K_b(x) = K_{-b}(x) \tag{17}$$

where b is any positive real number. When $\kappa < 0$ (i.e. $Q_F < 0$), the absolute value of κ, namely λ, is conveniently defined as:

$$[Q_F < 0:] \quad \lambda \equiv \frac{Q_{FN}}{4\pi\alpha} = -\kappa \tag{18}$$

in which $Q_{FN} = -Q_F$ is the magnitude of *inward* radial flow rate per unit length of cavity. The quantity λ characterizes the relative importance of the convection and conduction processes for inwards radial flow. It is used extensively herein and, because of equation (17), it can be used for the order of the Bessel function in the general solution equation (15), which becomes:

$$[\kappa < 0:] \quad \bar{T} = B_1 K_\lambda(r\beta) r^\kappa + T_0/s. \tag{19}$$

4. PRESCRIBED CONSTANT TEMPERATURE T_a AT THE CAVITY SURFACE

When a constant temperature T_a is maintained at the cavity surface, the Laplace transform of this boundary condition is T_a/s and evaluation of the constant B_1 in equations (15) and (19) leads to a composite result for both outward and inward seepage, as:

$$\bar{T} - (T_0/s) = (T_a - T_0) \frac{K_\kappa(r\beta)}{s K_\kappa(a\beta)} R^\kappa$$

$$\equiv (T_a - T_0) \frac{K_\lambda(r\beta)}{s K_\lambda(a\beta)} R^\kappa \tag{20}$$

where R is a non-dimensional radius defined as

$$R \equiv r/a. \tag{21}$$

By inspection, the sign of κ in equation (20) influences the transformed temperature only through the term R^κ, which is merely a constant in the integral of the inverse Laplace transform (equation (16)). Thus, the time-dependent part of the temperature distribution is, in this case, independent of whether the seepage is radially outwards or inwards.

Performing the inverse Laplace transform of equation (20) using equation (16) with the contour defined in Fig. 2, the temperature distribution can be written as:

$$T_T^* = \left[\left(\frac{1}{R}\right)^{|\kappa|} + \frac{2}{\pi} \int_0^\infty e^{-\alpha w^2 t} \right.$$

$$\left. \times \frac{J_\lambda(wr) Y_\lambda(wa) - Y_\lambda(wr) J_\lambda(wa)}{J_\lambda^2(wa) + Y_\lambda^2(wa)} \frac{dw}{w} \right] R^\kappa \tag{22}$$

where T_T^* is a non-dimensional temperature defined as

$$T_T^* \equiv \frac{(T - T_0)}{(T_a - T_0)} \tag{23}$$

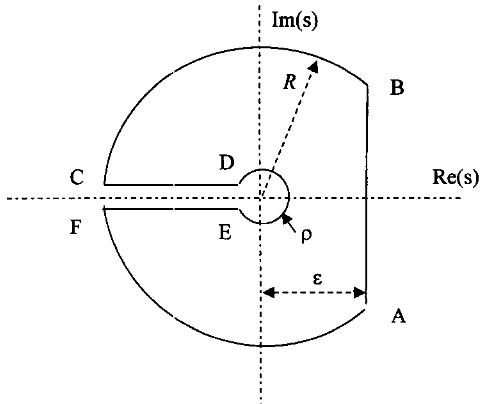


Fig. 2. Integration contour in the complex s plane.

the subscript T denoting the prescribed temperature boundary condition. Then $T_T^* = 1$ at the cavity surface and $J_\lambda(x)$ and $Y_\lambda(x)$ are Bessel functions of the first and second kinds of order λ. The parameter w is simply an integration variable. In ref. [10], a result is obtained corresponding to the special case of equation (22) when κ is positive, i.e. the fluid flow is radially outwards. When $\kappa = \lambda = 0$, equation (22) reduces to the solution for pure conduction given in ref. [10].

In principle, equation (22) is a general solution for the constant cavity temperature case. In practice, however, the integral can be evaluated analytically only for certain particular values of the flow parameters κ and λ. For other values, a numerical method of solution is necessary. One option is to perform the inverse Laplace transform of equation (20) numerically. Another is to evaluate the integral in equation (22) numerically; a numerical procedure similar to that outlined in Gemant [14] is satisfactory.

4.1. *Outward fluid flow, i.e. $\kappa > 0$*

4.1.1. *Transient conditions.* In the special case of $\kappa = 0.5$, the Bessel functions are of the spherical kind and the integral in equation (22) can be evaluated exactly. Introducing the non-dimensional quantities:

$$R^* = \frac{(R-1)}{2\sqrt{\tau}} \quad \text{and} \quad \tau = \frac{\alpha t}{a^2} \tag{24}$$

the temperature distribution is

$$[\kappa = 0.5:] \quad T_T^* = \operatorname{erfc}(R^*) \tag{25}$$

where erfc(x) is the complementary error function. This result is illustrated in Figs. 3(a and b) which show, respectively, the variation of temperature with time τ at a radius $R = 1.5$ and the variation with radius R at the time $\tau = 100$. Only the curves with $\kappa > 0$ are applicable to outward seepage. At any particular radius, steady-state conditions are approached more rapidly in the presence of seepage than when $\kappa = 0$. Likewise, at any particular time, the radius of influence of the cavity is decreased by the seepage.

4.1.2. *Steady-state conditions.* When $\kappa > 0$, the first term within the square brackets on the right-hand side of equation (22) reduces to unity when multiplied by R^κ. The second term is found to be negative and to tend to zero as time tends to infinity. Thus, the steady-state condition is simply:

$$[Q_F \geqslant 0:] \quad T_T^* = 1 \tag{26}$$

i.e. the temperature is uniform throughout the medium.

4.2. *Inward fluid flow, i.e. $\kappa < 0$*

4.2.1. *Transient conditions.* For the particular cases of $\lambda = -\kappa = 0.5$ and $\lambda = -\kappa = 1.5$, equation (22) can be integrated analytically, giving:

$$[\lambda = -\kappa = 0.5:] \quad T_T^* = [\operatorname{erfc}(R^*)]R^{-1} \tag{27}$$

and

$$[\lambda = -\kappa = 1.5:] \quad T_T^* = [\operatorname{erfc}(R^*) + (R-1)\exp(R-1+\tau)\operatorname{erfc}\{R^*+\sqrt{\tau}\}]R^{-3}. \tag{28}$$

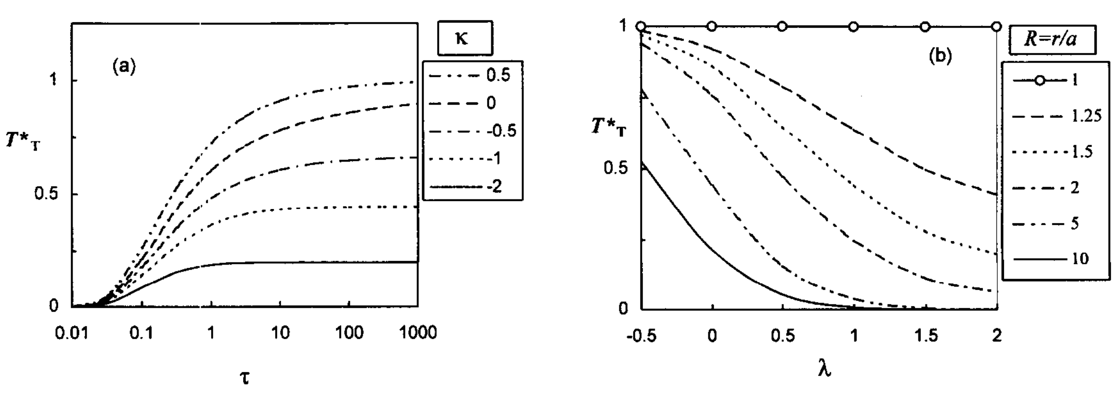


Fig. 3. (a) Influence of inwards and outwards seepage on temperature histories (constant cavity surface temperature, $R = 1.5$); (b) influence of radial position and seepage on temperature distributions (constant cavity surface temperature, $\tau = 100$).

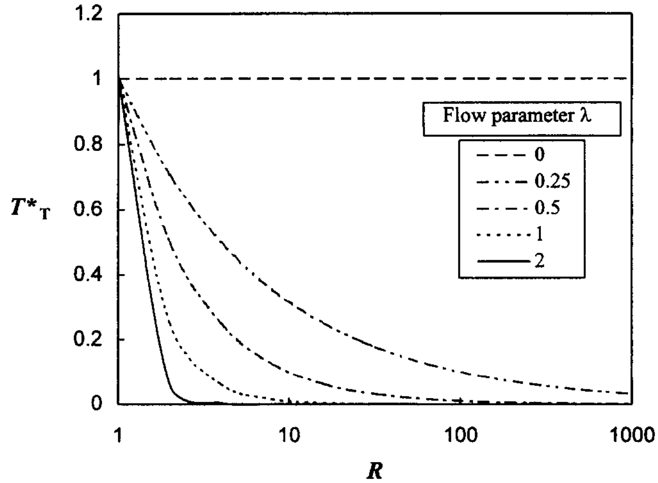


Fig. 4. Influence of inwards seepage on the steady-state temperature distribution (constant cavity surface temperature).

For other values of κ, equation (22) must be evaluated numerically.

In Fig. 3(a), it is seen that increase of seepage rate reduces the time required to approach the steady-state condition, which is 1 for $\kappa \geqslant 0$ and $1.5^{-2\lambda}$ for $\kappa < 0$ ($\lambda > 0$). For the particular radius shown ($R = 1.5$), steady conditions effectively prevail at $\tau = 100$ when $\lambda = 1$ (i.e. $\kappa = -1$), whereas they still do not exist at $\tau = 1000$ when $\lambda = \kappa = 0$. In the latter case, the time scales required to approach steady state can be so large that the conditions at moderate radii are always unsteady for practical purposes. Figure 3(b) shows temperature as a function of inflow rate λ at a time ($\tau = 100$) such the conditions are already close to steady state for the inflows ($\kappa < 0$), but they are far from the uniform steady-state temperature in the case of zero or outward flow ($\kappa \geqslant 0$). For $\lambda \geqslant 1.0$, the effective radius of influence is only about $5a$.

As a physical example, consider an 8 m diameter tunnel in sandstone. Using a typical value for α, it emerges that a value of 0.25 for λ corresponds to a seepage rate of about 4 l h^{-1} m^{-1} of tunnel. About 12 years would be required to attain an effective steady state compared with around 500 years in the absence of seepage.

4.2.2. *Steady-state conditions.* For inward seepage, $\kappa = -\lambda$ and $|\kappa| = \lambda$ and the first term in square brackets on the right-hand side of equation (22), when multiplied by $R^{-\lambda}$ becomes $R^{-2\lambda}$, while the integral term becomes zero in the limit as time tends to infinity. The steady-state temperature distribution is, thus:

$$T_T^* = R^{-2\lambda}. \tag{29}$$

This result, which could have been obtained by direct integration of equation (8), with $\sigma\,\partial T/\partial t = 0$, shows that with inward flow (unlike the outward flow case) a limited region of influence can occur at the steady state. Figure 4 shows the extreme sensitivity of the temperature distribution to inward seepage. At $r = 10a$ and $\lambda = 0.25$, the temperature is only 31% of that with no seepage. A further four-fold increase of seepage rate results in effectively negligible temperature at this position.

4.3. *Heat flux at the cavity surface*

4.3.1. *Transient conditions.* Using equations (9) and (22) and setting $R = 1$, the heat flux at the cavity surface due to heat conduction is found to be:

$$(q_T^*)_{R=1} = G + \frac{4}{\pi^2}\int_0^\infty e^{-\alpha w^2 t}\frac{dw}{w[J_\lambda^2(wa)+Y_\lambda^2(wa)]} \tag{30}$$

where the non-dimensional flux q^* is defined as

$$q_T^* \equiv \frac{aq_a}{k(T_a - T_0)} = -\frac{dT_T^*}{dR}. \tag{31}$$

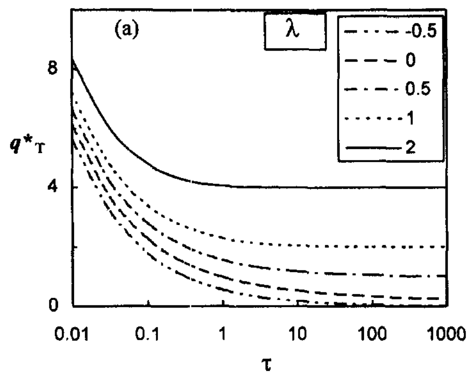
The parameter G in equation (30) is zero when $\kappa \geqslant 0$ and equals 2λ when $\kappa < 0$. For the particular cases of $\lambda = 0.5$ and 1.5, the heat flux at the cavity surface can be determined from equations (9), (31), (27) and (28), giving:

$$[\lambda = -\kappa = 0.5:] \quad (q_T^*)_{R=1} = 1 + \frac{1}{\sqrt{\pi\tau}} \tag{32}$$

and

$$[\lambda = -\kappa = 1.5:] \quad (q_T^*)_{R=1} = 3 + \frac{1}{\sqrt{\pi\tau}} - e^{\tau}\,\mathrm{erfc}(\sqrt{\tau}). \tag{33}$$

For integer values of λ, the integral in equation (30) can be evaluated numerically. The dependence of the surface heat flux on time and on the rate of seepage are illustrated in Figs. 5(a) and (b). At any particular



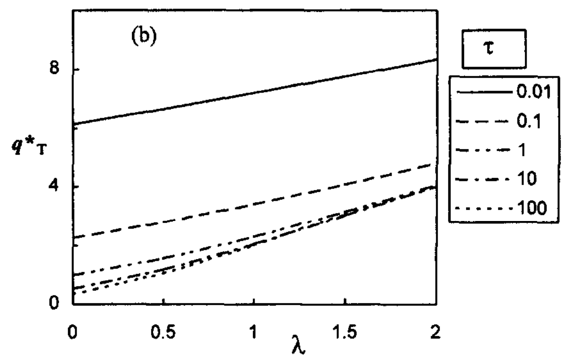


Fig. 5. (a) Evolution of heat flux at the cavity surface—dependence on inwards and outwards seepage (constant cavity surface temperature, $R = 1$); (b) influence of inwards seepage on heat flux at the cavity surface (constant cavity surface temperature; $R = 1$).

time, the heat flux increases with increasing inwards seepage; it decreases with increasing outwards seepage. The greatest relative influence on the conduction heat flux is at large times. When $\lambda = 0$, the expression in equation (30) reduces to that for pure conduction.

4.3.2. *Steady-state conditions.* Since when $t \to \infty$, the integral in equation (30) vanishes, it follows that G is the steady-state cavity heat flux. Thus, for outward or zero flow ($\kappa \geqslant 0$), the cavity flux vanishes at the steady state, whereas for inward flow ($\kappa < 0$), there is a steady-state flux.

5. PRESCRIBED HEAT FLUX AT THE CAVITY SURFACE

When a constant conduction heat flux per unit length $q_a = Q_h/2\pi a$ is maintained at the cavity surface, the boundary condition is:

$$-k\left(\frac{dT}{dr}\right)_{r=a} = q_a. \tag{34}$$

The constant B_1 in equations (15) or (19) can be found using the Laplace transform of equation (34). The transformed temperature distribution is then known.

5.1. *Outward fluid flow*

5.1.1. *Transient conditions.* The transformed temperature distribution in this case is:

$$\bar{T}^*_H = \frac{K_\kappa(r\beta)R^\kappa}{s\beta a K_{\kappa-1}(\beta a)} \tag{35}$$

in which $\bar{T}^*_H$ is the transform of the non-dimensional temperature T^*_H defined as

$$T^*_H = \frac{k(T-T_0)}{aq_a}. \tag{36}$$

After inverse Laplace transformation, the temperature distribution is:

$$T^*_H = \left[Vt + \frac{2}{\pi a}R^\kappa \int_0^\infty (1-e^{-\alpha w^2 t}) \times \frac{J_\kappa(wr)Y_{\kappa-1}(wa) - Y_\kappa(wr)J_{\kappa-1}(wa)}{w^2[J^2_{\kappa-1}(wa)+Y^2_{\kappa-1}(wa)]}\,dw\right] \tag{37}$$

where

$$V \equiv 0 \quad \text{when } 0 \leqslant \kappa \leqslant 1$$

$$V \equiv 2\alpha(\kappa-1)/a^2 \quad \text{when } \kappa > 1. \tag{38}$$

when $\kappa = 0$, equation (37) reduces to the result for zero seepage given by Carslaw and Jaeger in ref. [10]. For integer values of κ, equation (37) can be evaluated numerically. For the particular case of $\kappa = 0.5$, the Bessel functions are spherical and the temperature distribution can be found analytically as:

$$T^*_H = 2\sqrt{(\tau/\pi)}\exp[-(R^*)^2] - (R-1)\,\text{erfc}(R^*) \tag{39}$$

which reduces at the cavity surface to

$$(T^*_H)_{R=1} = 2\sqrt{\tau/\pi}. \tag{40}$$

These results show that for $\kappa = 0.5$, the temperature at any point in the porous medium continues to increase without limit in time.

5.1.2. *Steady-state conditions.* By direct integration of equation (8) with $\sigma\,\partial T/\partial t = 0$, the temperature distribution for $\kappa > 0$ is found to be of the form:

$$T = AR^{2\kappa} + \frac{B}{2\kappa} \tag{41}$$

where A and B are constants. Thus, the temperature cannot be bounded for $\kappa > 0$ unless T is uniform. For the boundary condition of equation (34) it follows that the only possible steady state is the trivial one of uniform temperature equal to the initial temperature T_0. This can also be demonstrated by considering the form of equation (35) as $s \to 0$, which by the properties of the Laplace transform, corresponds to conditions as $t \to \infty$. For $\kappa > 1$, for example, $\bar{T}^*_{\mathrm{H}} \propto s^{-2}$ which implies that $T^*_{\mathrm{H}} \propto t$, so that there is no steady state. Physically, as long as heat is supplied at the cavity, the temperature in the porous medium will increase when there is outward seepage (or no seepage as shown in ref. [10]).

5.2. *Inward fluid flow*

5.2.1. *Transient conditions.* When the seepage is inwards, i.e. $\kappa < 0$ and $\lambda > 0$, the transformed temperature distribution is:

$$\bar{T}^*_{\mathrm{H}} = \left(\frac{K_\lambda(r\beta)}{as\beta K_{\lambda+1}(a\beta)}\right)R^{-\lambda} \tag{42}$$

and the temperature distribution is

$$T^*_{\mathrm{H}} = \left[\frac{1}{2\lambda R^\lambda} + \frac{2}{\pi a}\int_0^\infty \mathrm{e}^{-\alpha w^2 t} \times \frac{J_\lambda(wr)Y_{\lambda+1}(wa) - Y_\lambda(wr)J_{\lambda+1}(wa)}{w^2[J^2_{\lambda+1}(wa) + Y^2_{\lambda+1}(wa)]}\mathrm{d}w\right]R^{-\lambda}. \tag{43}$$

The Bessel functions in this equation are of the spherical kind when $\lambda = 0.5$, and the temperature can then be obtained as:

$$T^*_{\mathrm{H}} = [\mathrm{erfc}(R^*) - \mathrm{e}^{R-1+\tau}\,\mathrm{erfc}(R^* + \sqrt{\tau})]R^{-1}. \tag{44}$$

At the cavity surface, where $R = 1$, $R^* = 0$ and $\mathrm{erfc}(R^*) = 1$, the temperature is:

$$(T^*_{\mathrm{H}})_{R=1} = 1 - \mathrm{e}^\tau\,\mathrm{erfc}(\sqrt{\tau}). \tag{45}$$

Figure 6(a) shows temperature distributions at successive times for the particular case of $\lambda = 1.0$. The temperature at the cavity surface reaches approximately 50% of its long-term steady value when τ is as small as 0.1. It reaches 90% of the steady-state value when $\tau \approx 2$. The radial influence of the cavity is relatively small at all times. For instance, the heat flux at radii greater than $R = 2.5$ never exceeds 20% of the steady-state value at the cavity surface. Figure 6(b) shows the influence of the inwards fluid flow rate on the temperature at the cavity surface at various times. At sufficiently small times, the influence of the seepage is very small, whereas at large times, it is quite strong.

5.2.2. *Steady-state conditions.* As $t \to \infty$, equation (43) reduces to a non-trivial steady state:

$$T^*_{\mathrm{H}} = \left(\frac{1}{2\lambda}\right)R^{-2\lambda} \tag{46}$$

while the temperature at the cavity surface becomes

$$(T^*_{\mathrm{H}})_{R=1} = \frac{1}{2\lambda}. \tag{47}$$

From Fig. 6(b), it is seen that steady state is approached rapidly when $\lambda > 1$, but much larger times are required at small seepage rates. It is never achieved at $\lambda = 0$ (no seepage). The temperature distributions at the steady state for a range of the parameter λ are shown in Fig. 7. As may be seen by comparing equations (46) and (29), the curves in Fig. 7 are scaled by $1/(2\lambda)$ compared with Fig. 4. The particular case of $\lambda = 0.5$ is the same in both figures.

6. APPROXIMATE SOLUTIONS FOR INWARD RADIAL FLOW

All exact analytical solutions presented herein are for seepage rates corresponding to half integer order Bessel functions. All other results have been obtained by numerical evaluation of quadratures such as in equation (22) and are for seepage rates corresponding to integral or half integral orders of the Bessel functions. However, short- and long-time approximations valid for all λ can be obtained in most cases by using

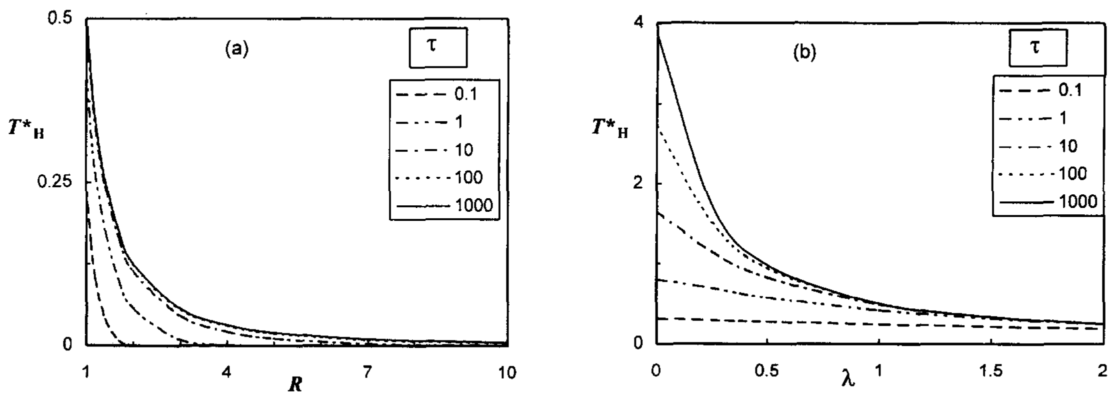


Fig. 6. (a) Evolution of temperature distributions with inwards seepage (constant cavity surface heat flux, $\lambda = 1$); (b) evolution of the influence of inwards seepage on the cavity surface temperature (constant cavity surface heat flux, $R = 1$).

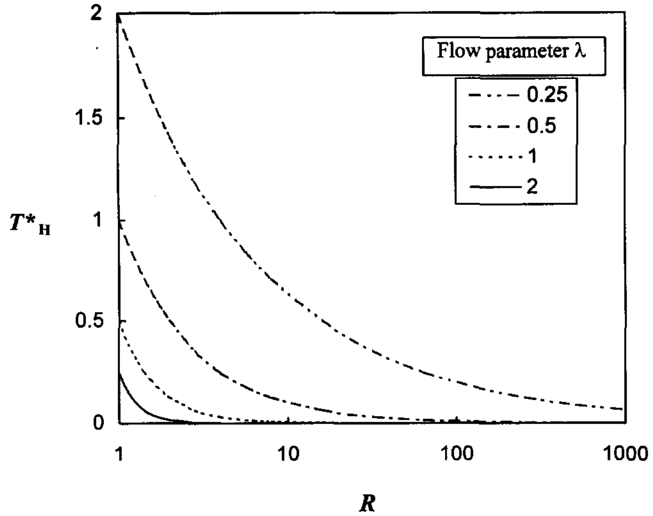


Fig. 7. Influence of inwards seepage on the steady-stated temperature distribution (constant cavity surface heat flux).

limiting forms (for small and large arguments) of the Bessel functions appearing either in the transform solutions, for example, equation (20), or in the integrands of the inverse transform solutions, for example, equation (22).

6.1. *Prescribed constant temperature* T_a *at the cavity surface*

6.1.1. *Approximate small-time solution.* As can be seen from equation (10), large values of s in the Laplace transform imply that only small values of time t contribute significantly to the transform value. It follows that expressions valid for small times can be derived using approximations that are valid for large arguments of the modified Bessel functions in equation (20). For large values of x, the leading terms of the asymptotic expansion of the modified Bessel function $K_\lambda(x)$ have the form:

$$K_\lambda(x) \approx \sqrt{\frac{\pi}{2x}}\mathrm{e}^{-x}\left[1+\frac{4\lambda^2-1}{8x}\right]. \qquad (48)$$

Using equations (20) and (48), an approximate solution for the temperature distribution valid at small times is:

$$T_T^* \approx [\![\mathrm{erfc}(R^*)+(R-1)\exp[\zeta(R-1)+\zeta^2\tau] \times \mathrm{erfc}(R^*+\zeta\sqrt{\tau})]\!]R^{-(\lambda+1.5)} \qquad (49)$$

where

$$\zeta = (4\lambda^2-1)/8. \qquad (50)$$

This expression is compared with the numerical evaluation of equation (22) in Fig. 8(a) for the particular case of $\lambda = 1$. The agreement is satisfactory for τ less than about 1.0. Although equation (49) is approximate for most values of λ, it is exact when λ is equal to 0.5 and 1.5 (indeed, it is then exact for *all* times). In these cases, the asymptotic expansions of the Bessel functions terminate after a finite number of terms, giving expressions that are identical to the exact form of these spherical functions.

Using equations (9) and (49), the conduction *heat flux* at the cavity surface at small times is obtained as:

$$(q_T^*)_{R=1} \approx (\lambda+1.5)+\frac{1}{\sqrt{\pi\tau}}-\exp(\zeta^2\tau)\,\mathrm{erfc}(\zeta\sqrt{\tau}) \qquad (51)$$

which coincides with the exact solutions for $\lambda = 0.5$ and 1.5 [equations (32) and (33)]. For $\lambda = 1$, equation (51) is compared with the numerical evaluation of equation (30) in Fig. 8(b) and is seen to give satisfactory results for τ smaller than about 1.

6.1.2. *Approximate large-time solution.* By inspection of equation (10), the use of a small value of s in the Laplace transform will ensure that the result is dominated by the contribution of large values of t. Thus, an approximate temperature distribution valid at long times can be obtained by replacing the modified Bessel functions in equation (20) by a suitable approximation valid for small arguments. In the case that λ is a positive real number other than an integer, an appropriate expansion is:

$$K_\lambda(x) \approx \frac{\pi}{2\sin\lambda\pi}\left(\frac{2}{x}\right)^\lambda \left\{\left[\frac{1}{\Gamma(1-\lambda)}-\frac{1}{\Gamma(1+\lambda)}\left(\frac{x}{2}\right)^{2\lambda}\right] + \left(\frac{x}{2}\right)^2\left[\frac{1}{\Gamma(2-\lambda)}-\frac{1}{\Gamma(2+\lambda)}\left(\frac{x}{2}\right)^{2\lambda}\right]\right\} \qquad (52)$$

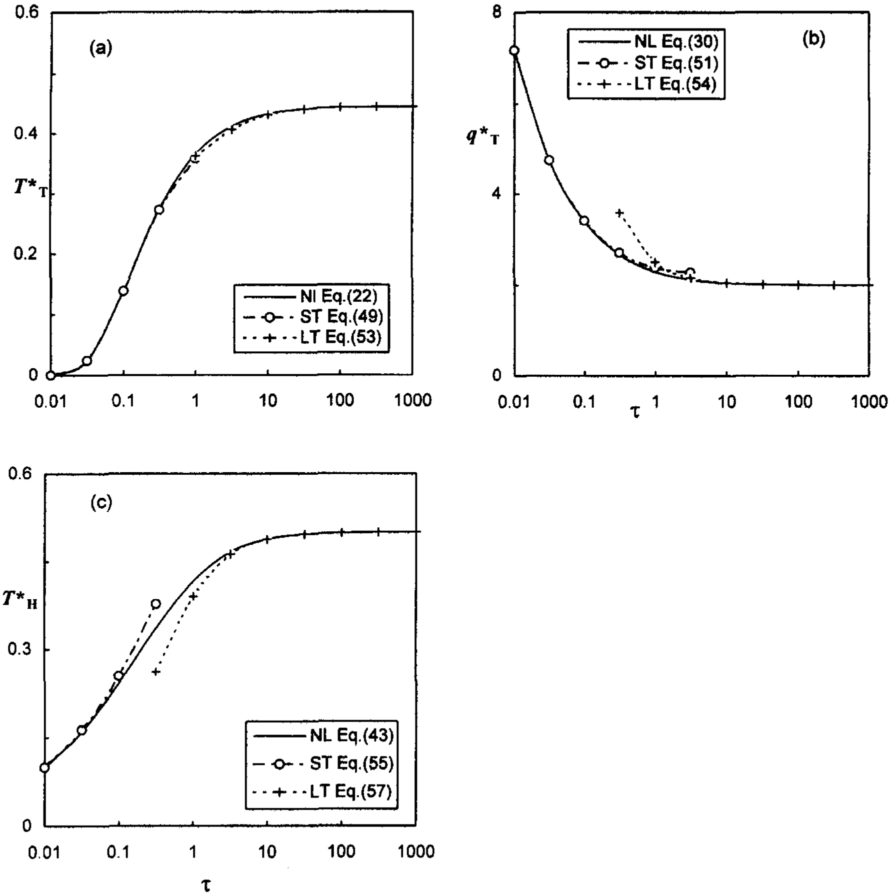


Fig. 8. NL, and ST, LT denote numerical, and short- and long-term approximate solutions respectively. (a) Short- and long-term approximations for the cavity surface temperature with inwards seepage—comparisons with exact solution (constant cavity surface temperature, $\lambda = 1.0$, $R = 1.5$); (b) short- and long-term approximations for the heat flux at the cavity surface with inwards seepage (constant cavity surface temperature, $\lambda = 1.0$, $R = 1$); (c) short- and long-term approximations for the cavity surface temperature with inwards seepage (constant cavity surface heat flux, $\lambda = 1.0$, $R = 1$).

where $\Gamma(\lambda)$ is the gamma function. This expression has been derived from the definition of $K_\lambda(x)$ in terms of $I_\lambda(x)$ and $I_{-\lambda}(x)$ using a series expansion of the latter functions. It is applicable for $\lambda > 0$, but excludes $\lambda = 1$ and 2 for which standard expressions are available in Abramowitz and Stegun [15].

By substituting equation (52) (or an expression for $\lambda = 1$ or 2) into equation (20) and taking the inverse Laplace transform, the approximate temperature distribution for $\lambda > 0$ is of the form:

$$T_T^* \approx \left[1 - \frac{(R^{2\lambda}-1)}{\Gamma(1+\lambda)(4\tau)^\lambda} + \frac{(R^{2\lambda+2}-1)}{\Gamma(2+\lambda)(4\tau)^{\lambda+1}}\right] R^{-2\lambda}. \tag{53}$$

The validity of this result for a larger integer value of λ has not been verified. At $\lambda = 1$, there is good agreement in Fig. 8(a) (for τ greater than about 3) with a numerical evaluation of equation (22). At zero seepage ($\lambda = 0$), equation (53) does not apply but suitable approximations are given in Ritchie and Sakakura [16].

The *heat flux* at the cavity surface at large times, determined from equations (9) and (53) is:

$$(q_T^*)_{R=1} \approx (2\lambda)\left[1 + \frac{1}{\Gamma(1+\lambda)(4\tau)^\lambda} - \frac{1}{\Gamma(2+\lambda)(4\tau)^{\lambda+1}}\left(1+\frac{1}{\lambda}\right)\right]. \tag{54}$$

Figure 8(b) compares this approximation, at $\lambda = 1.0$, to the numerical evaluation of equation (30), with satisfactory agreement for $\tau > 3$.

6.2. *Prescribed constant heat flux* q_a *at the cavity surface*

6.2.1. *Approximate small-time solution.* Equation (48) can be used in equation (42) to obtain a small-time approximation for the cavity surface temperature due to constant cavity surface heat flux q_a in the form:

$$(T^*_{\mathrm{H}})_{R=1} \approx \frac{1}{\eta}\{1-\exp(\eta^2\tau)\operatorname{erfc}(\eta\sqrt{\tau})\} + \left(\frac{2\zeta}{\eta}\right)\sqrt{\frac{\tau}{\pi}} - \frac{\zeta}{\eta^2}\{1-\exp(\eta^2\tau)\operatorname{erfc}(\eta\sqrt{\tau})\} \quad (55)$$

where

$$\eta = [(\lambda+1)^2-1]/8. \quad (56)$$

A comparison in Fig. 8(c), with the numerical evaluation of equation (43) for $\lambda = 1.0$, shows satisfactory agreement for $\tau < 0.1$. The case of $\lambda = 0$ when equation (55) is singular is considered in ref. [16].

6.2.2. *Approximate large-time solution.* Using equation (52) in equation (42), the approximation cavity surface temperature is:

$$(T^*_{\mathrm{H}})_{R=1} = \frac{1}{2\lambda}\left[1 - \frac{1}{\Gamma(1+\lambda)(4\tau)^{\lambda}} + \frac{1}{\Gamma(\lambda+2)(4\tau)^{\lambda+1}}\right]. \quad (57)$$

In Fig. 8(c), this result is compared with numerical integration of equation (43). The agreement is satisfactory for $\tau > 1$. The case of zero seepage ($\lambda = 0$) for which equation (57) is singular, is considered by Carslaw and Jaeger in ref. [10].

7. CONCLUSIONS

(1) Analytical solutions in the form of quadratures have been obtained for transient heat conduction radially to or from a cylindrical cavity in a permeable medium with steady radial fluid seepage outwards or inwards for the following thermal boundary conditions at the cavity surface: (i) a prescribed constant temperature; and (ii) a prescribed constant heat flux.

(2) Exact closed-form solutions have been found for these cases at certain particular seepage rates. Approximate analytical solutions have been found for all seepage rates at small time and at large time (approaching steady state). Numerical solutions have been obtained for all seepage rates at all times.

(3) With *zero* or *outwards* seepage and prescribed cavity surface temperature, the radial influence of the cavity increases indefinitely with time. Close to the cavity, the conditions approach a uniform steady state if the cavity surface temperature is constant. In contrast, in the case of prescribed cavity surface heat flux, the temperature increases without limit.

(4) With *inwards* seepage and prescribed cavity surface temperature or heat flux, the effective radial influence of the cavity is limited at small times for all seepage rates and at all times for large seepage rates. The effective radius of influence decreases with increasing seepage rate. Within this radius, the heat flux is significant at all times. The approach to steady state is progressively more rapid as the fluid flow rate increases.

(5) In the case of *inwards* seepage, a steady state is found to exist for both types of boundary condition at the cavity surface. Exact solutions have been found for all seepage rates. The thermal influence of the cavity is limited and decreases with increasing seepage rate.

(6) In the case of *zero* or *outwards* seepage and prescribed cavity surface temperature, the only possible steady-state condition is that of uniform temperature throughout the medium.

REFERENCES

1. Gringarten, A. C., Witherspoon, P. C. and Ohnishi, Y., Theory of heat extraction from fractured hot dry rock. *Journal of Geophysical Research*, 1975, **80**, 1120–1124.
2. Claesson, J. and Dunand, A., *Heat Extraction from the Ground by Horizontal Pipes*. Swedish Council for Building Research, Stockholm, 1983.
3. Iguchi, Y., Estimation of temperature in Seikan tunnel. *Quarterly Reports of the Railway Technical Research Institute, Japan*, 1985, **26**, 145–150.
4. McPherson, J. M., The analysis and simulation of heat flow into underground airways. *International Journal of Mining and Geological Engineering*, 1986, **4**, 165–196.
5. Kimura, S., Masuda, Y. and Hayashi, K., Efficient numerical method based on double porosity model to analyze heat and fluid flows in fractured rock formations. *JSME International Journal*, 1992, **35**, 395–399.
6. Bau, H., Convective heat losses from a pipe buried in a semi-infinite porous medium. *International Journal of Heat and Mass Transfer*, 1984, **27**, 2047–2056.
7. Nicolson, J. W., A problem in the theory of heat conduction. *Proceedings of the Royal Society of London*, 1921, **A100**, 226–240.
8. Goldstein, S., Some two-dimensional diffusion problems with circular symmetry. *Proceedings of the London Mathematical Society*, 1931, series 2, **34**, 51–88.
9. Carslaw, H. S. and Jaeger, J. C., Some two-dimensional problems in conduction of heat with circular symmetry. *Proceedings of the London Mathematical Society*, 1940, series 2, **46**, 361–388.
10. Carslaw, H. S. and Jaeger, J. C., *The Conduction of Heat in Solids*, 2nd edn. Oxford University Press, London, 1959.
11. Cheng, P., Heat transfer in geothermal systems. *Advances in Heat Transfer*, 1978, **14**, 1–105.
12. Poulikakos, D. and Bejan, A., Natural convection in a porous layer heated and cooled along one vertical side. *International Journal of Heat and Mass Transfer*, 1984, **27**, 1879–1891.
13. Nield, D. A. and Bejan, A., *Convection in Porous Media*, 1st edn. Springer, Berlin, 1992.
14. Gemant, A., Transient temperatures around heating pipes maintained at constant temperature. *Journal of Applied Physics*, 1946, **17**, 1076–1081.
15. Abramowitz, M. and Stegun, I. A., *Handbook of Mathematical Functions*, 1st Dover edn. Dover, New York, 1965, p. 378.
16. Ritchie, R. H. and Sakakura, A. Y., Asymptotic expansions of solutions of the heat conduction equation in internally bounded cylindrical geometry. *Journal of Applied Physics*, 1956, **27**, 1453–1459.